Claudius Ptolemy's Law of Refraction

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Emus tenui aqua tegitur et fracti speciem reddit …

"An oar is covered with shallow water and gives the appearance of being broken." That's Lucius Annaeus Seneca, noted Roman philosopher and author, writing in his book, "*Natural Questions*," circa 63 C.E.[1] Seneca's is an early observation that light passing into water is refracted, but it's not the earliest. Titus Lucretius Caro had expressed much the same thought about a century earlier[2] in his philosophical poem, "*De Rerum Natura*."* Lucretius's poem was an interpretation of Epicurean philosophy, so the observation could easily be centuries older. It's hard to imagine those seagoing Greeks not noticing the way the line of a submerged oar seems to "break" at the interface between air and water.

This is how the science of optics usually found expression in the ancient world: a single observation, rather vague, qualitative rather than quantitative. Some observations, by contrast, were surprisingly accurate and modern, seemingly far ahead of their time. They weren't, of course; they were part and parcel of a tradition of careful and exact measurement that was largely forgotten before the Fall of Rome and for the most part has not been translated into modern English. Examples are well known to some classical scholars and to historians of science, but not to the general public or even to most scientists. When you stumble upon a gem such as "*The Sand-Reckoner*" by Archimedes (which has a discussion of extremely large numbers and estimates the number of grains of sand needed to fill the universe), you are astonished by its clarity and modernity.

One such piece of ancient science I stumbled across was Claudius Ptolemy's determination of the law of refraction. I had learned, as I think every optics student has, that the law of refraction was discovered by Dutch mathematician Willebrord van Roijen Snell in 1621 and first published in Christiaan Huygens 1703 book, "*Dioptrica*." The law was not published by Snell during his lifetime and was independently discovered by René Descartes; it is thus known in France as Descartes's law. By means of this powerful law, Descartes was able to calculate the location of the rainbow and Newton was able to derive the laws of imaging with lenses. Because of the careful experimental work of Snell, Descartes and others, modern optics took root in the 17th century.[3]

Mathematician, astronomer and physicist Claudius Ptolemy performed a series of careful experiments in the first century C.E. to determine the rules of refraction.

So I was amazed to discover that mathematician, astronomer and physicist Claudius Ptolemy had already performed a series of careful experiments in the first century C.E. to determine the rules of refraction. What's more, Ptolemy's was a quantitative study, and a startlingly modern one at that, performed just as it is today in many undergraduate optics labs.

Ptolemy constructed a straightforward goniometer, marking off the degrees along the edge and placing the center at the interface between two dielectric media. He examined refraction at the interface between air and water, between glass and air, and between glass and water. He varied the angle of incidence between 10 degrees and 80 degrees in units of 10 degrees (measured from the normal to the interface) and measured the corresponding angle of refraction. So why didn't the world get the law of refraction fifteen hundred years before Snell and Descartes?

I first learned of Ptolemy's work in C.B. Boyer's wonderful book, "*The Rainbow: From Myth to Mathematics*."[4] In it, Boyer notes that George Sarton, in his "*Introduction to the History of Science*,"[5] called Ptolemy's work "the most remarkable experimental research of antiquity." Ptolemy's work certainly does follow the model of experimental science we've been brought up to revere. He made an observation (the apparently "broken" oar), hypothesized a cause (change of angle at the water-air interface), arranged an experiment in exemplary fashion, made his observations and came up with a result. So why didn't history follow the rest of the script, giving us modern optics a millennium and a half earlier?

To listen to some of the critics, you'd think that Ptolemy *didn't* get it right at all. A look at Internet sites that deal with the topic imply that Ptolemy's numbers were off by quite a bit, and that his big assumption was that for small angles, the angle of incidence is proportional to the angle of refraction. Even Boyer, after presenting the evidence, notes that "A closer glance … suggests that there was less experimentation involved in it than originally was thought … As in other portions of Greek science, confidence in mathematics was here greater than that in the evidence of the senses, although

* The title is sometimes translated as, "On the Nature of the Universe," but it could as easily be rendered, "The Way Things Are."

the value corresponding to 60° agrees remarkably well with experience."

Someone reading the critics might be led to conjecture: "Ptolemy had a good idea but didn't follow through well. He wasn't a good enough experimentalist, or he let his expectations rule the observations. He got the figure right on about 60 degrees, but not on the other values."

But such conclusions would be incorrect. I submit, as evidence, Fig. 1, in which Ptolemy's results for the air-water interface are plotted alongside modern results obtained from Snell's law. The 60-degree data point is not, in fact, the sole intersection with reality: Ptolemy's data agree with modern findings to within about 0.5 degrees for all values except 80 degrees. (In fact, the best agreement isn't at 60 degrees but at 70 degrees.)

We get similar results if we plot Ptolemy's results against modern results for the air-glass or the water-glass interface. (I assumed $n = 1.5$ for the refractive index of glass—the first refuge of the optical engineer.) All things considered, Ptolemy's results are pretty darned good. So why does everyone say they're bad?

One reason must be that his work has not been widely disseminated, so most people haven't seen it firsthand. The figures for the air-water interface are in Boyer, but the other figures I obtained from A. Mark Smith's "*Ptolemy and the Foundations of Ancient Mathematical Optics*."[7] Smith, in turn, took them from his own translation of Ptolemy's "*Optics*." As far as I can tell, Smith's is the first translation of this important work into English. It appeared as recently as 1996.[8]

Several writers take Ptolemy to task for not having determined the correct relationship between the sines of the angles of incidence and refraction. This is somewhat ironic, since Ptolemy compiled what is essentially a widely used table of sines. What's more, it's by no means obvious that the sines of the angles would be the functions to have a linear relationship. Moreover, Ptolemy's tables did not give the sines of the angles directly, but rather the lengths of the chords of those angles, measured on a circle having a radius of 60 units. The chord of an angle is proportional to the sine of half the angle, and this makes the relationship even less obvious.[9]

Furthermore, Ptolemy was very clearly taken by another feature of his tables of angles. In all three cases (air-water interface, air-glass interface and water-glass interface) the second differences between the angles of refraction are constant. In other words, while the angle of incidence increases by 10 degrees each time, the intervals between the angles of refraction are decreasing (8 degrees, 15.5 degrees, 22.5 degrees, 29 degrees differ by, successively, 7.5 degrees, 7 degrees and 6.5 degrees). The difference between the intervals is always -0.5 degrees. Furthermore, Ptolemy found the second difference to be the same in all three cases —it's always -0.5 degrees. This amounts to saying that the relationship between the angles of incidence and refraction, measured in degrees, is quadratic. Happily, the equations assume a relatively simple form. For air-water, the relationship is:

$$\theta_{refract} = \frac{33}{40}\theta_{incident} - \frac{1}{400}\theta^2_{incident}$$

For air-glass, the linear coefficient becomes 29/40, and for water-glass it becomes 39/40. The quadratic coefficient, of course, is always (-1/400).

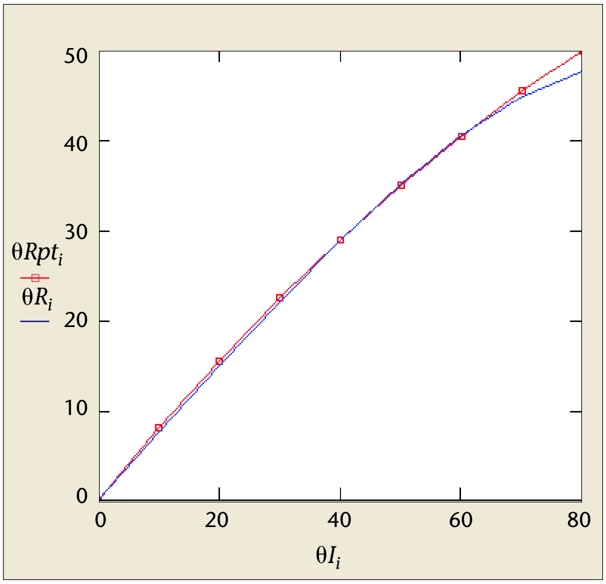
Two errors are involved. Ptolemy rounded his numbers, almost always to the nearest half of a degree. But not always, especially for the refraction corresponding to 80-degree incidence. He evidently saw the beginnings of a quadratic pattern and allowed it to shape his reported observations. As has been noted by others,[10] Ptolemy's reported angles of refraction aren't really "raw" data: they've been smoothed and massaged.

Smith suggests that Ptolemy was "conditioned" by his work in astronomy to see a pattern of constant second differences,[11] so it's not surprising that he noticed what he thought was the same pattern in optics. If you use the correct values and Snell's law, of course, you find that the second difference is not a constant -0.5 deg^{-1}. The second derivative of the function varies considerably from 0 deg^{-1} at 0 degrees to about -1.5 deg^{-1} at 70 degrees for air-water.

Why did Ptolemy fail to discover Snell's law? Smith says that Ptolemy wasn't trying to answer the same questions: he was not seeking to explain the radiation of light, but the nature of sight.[12] It seems to me, however, that to explain one is to explain the other. Ptolemy had great successes elsewhere in his work, why did he fail here?

It seems clear to me that Ptolemy felt he had reached the correct result and decided to push no farther. It was not of central importance to him, despite the investment in time and equipment. Having satisfied himself with an answer, he moved on.

Ptolemy's result may not have been perfect, but it was, as shown above, not a bad result. It certainly provided concrete



Ptolemy's measurements for angle of refraction vs. angle of incidence at air-water interface (red) vs. predictions based on Snell's law (blue).

numbers for anyone wishing to pursue a mathematical study of refractive optics. With the advantage of two millennia of mathematics and physics, we can see that the function

$$\theta_{refract} = \sin^{-1}\left\{\frac{1}{n}\sin\left(\theta_{incident}\right)\right\}$$

ought to be expanded in a Taylor series, giving a linear term followed by a cubic, rather than by a quadratic, term. If you use only the first two terms of the expansion, however, your result won't be as good as Ptolemy's is with his two terms. So maybe it's not so bad to stick to his formulation.

What would have happened if someone had actually used Ptolemy's law to formulate geometrical optics? Well, the first thing you'd find is that the refractive indices don't come out quite right. Water's is 40/33, giving it a refractive index of 1.212 instead of 1.33. Glass has an index of 40/29 = 1.37 instead of 1.5. The results carry through to cause problems in later calculations. You can derive paraxial optics and end up with the same results we have today, including the lensmaker's formula for the power of a lens (since the assumptions of a linear relationship between angles of incidence and angles of refraction were expressed by Ptolemy for small angles). But if you were to "plug in" the above values you'd find that the incorrect refractive index leads to an error: the predicted power of the lens would be too low, the focal length too long.

In fact, we need not go to all that trouble to find a situation where Ptolemy's measurements will produce a discrepancy with experiment. We can consider an effect very similar to the ancient "broken oar" problem: the fact that a ring placed at the bottom of a basin appears to be closer when the basin is filled with water.[13] Using simple paraxial optics, you calculate that the apparent distance of the ring from the water's surface is $1/n$ times the distance it truly is. This experiment would be simple to carry out and the difference between an index of 1.21 and one of 1.33 would be speedily noted.

The problem is, of course, that no one actually did the experiment. Ptolemy never followed through on his work, and no one else carried it on. If anyone had pursued this tantalizing lead, they would undoubtedly have discovered the inconsistencies. That, of course, is the sort of thing that is necessary for good science: peer review and testing of hypotheses. Ptolemy's experiment resembles less a modern research paper than it does an undergraduate experiment: performed once, written up and not reviewed again. Had Ptolemy been forced to explain the discrepancies, he might even have produced the law of refraction as we know it, despite those cumbersome sines expressed as chords.

But until fairly recently, optics did not receive treatment as comprehensive as did other branches of mathematics and science. The evidence lies in the fragmentary state of our knowledge of ancient optics. We know that several books on the topic were penned, but few have survived and those that have are incomplete. For Ptolemy's own "*Optics*," we have only a 12th century Latin translation of an Arabic copy, and we lack Book I and the end of Book V.[14] And, as I noted, an English translation has only become available within the past decade. Ptolemy's ideas did not get fully corrected for the same reason they did not have a significant impact on the history of optics: they were not widely disseminated.

As Neuegebauer notes, Ptolemy's work had its greatest influence through the Arabic students of optics, who knew of it indirectly through the work of Ibn al-Haitham (better known as Alhazen). Even with this audience, however, it did not find full expression. Among the lost parts of Ptolemy's "*Optics*" is his explanation of the rainbow. The Arabic opticists Qutb al-Din al-Shirazi (1236-1311) and his student Kamal al-Din al-Farisi (circa 1320) both studied the rainbow, producing surprisingly modern results. They modeled the raindrop as a sphere of water and experimented with a glass sphere filled with water, following the beam of light as it refracted, reflected once inside, then refracted upon leaving the drop. They correctly explained both the primary and secondary rainbow in this way, and made the first observation of a tertiary rainbow. At almost precisely the same time, a French-German monk named Theodoric of Freibourg was working along the same lines in Europe. His drawings have come down to us (al-Shirazi's and al-Farisi's, sadly, have not), and his pictures of a primary or a secondary rainbow could have been taken from a modern optics text, or from Descartes' work on the rainbow.

Descartes, you may recall, used Snell's law to imagine the trajectories of rays entering a drop of water at different distances from the center and, noting that the rainbows occurred at a maxima or a minima in the angle of refraction, calculated for the first time the angle of the rainbow. If they had been armed with Ptolemy's formula, al-Farisi, al-Shirazi or Theodoric, who knew the path of the ray through the drop, could have done the same 300 years earlier. Moreover, they would have obtained the correct result. Unlike the paraxial optics examples I noted above, which depend on the values at small angles (where Ptolemy's insistence on constant second differences led him into error), the rainbow angle is calculated on angles where Ptolemy's results are almost perfectly correct (despite his having fudged the data). The quadratic formula you derive from Ptolemy's data predicts that the rainbow lies at 42 degrees from the anti-solar point. The modern geometrical optics result, using Snell's law, predicts 42.5 degrees.[15]

In other words, we came extremely close to having a correct theoretical explanation for the rainbow several centuries in advance of its actual discovery, and only missed it (arguably) because Ptolemy's book didn't quite last long enough or wasn't popular enough to reach the proper hands.

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References

1. Book I, section 3.9. I quote both the Latin and the translation from the Loeb Classical Library edition of "*Seneca: Natural Questions (Naturales Quaestiones)*," Books I-III, trans. T. H. Corcoran, (Harvard Univ. Press, 1971), 38-9.
2. Lucretius's observation of the apparent bending of the oar (and of the rudder as well) appears in Book IV, line 437. On page 144 of the Penguin translation by R.E. Latham, this is rendered as: "So much of the oars as projects above the waterline is straight, and so is the upper part of the rudder. But all the submerged parts appear refracted and wrenched round in an upward direction and almost as though bent right back so as to float on the surface." (Penguin Books, 1951).
3. See, for instance, page 3 of E. Hecht's "*Optics*," (2nd ed., Addison-Wesley, 1987) "… this was one of the great moments in optics. By learning precisely how rays of light are redirected on traversing a boundary between two media, Snell in one swoop swung open the door to modern applied optics."
4. C. B. Boyer, "*The Rainbow: From Myth to Mathematics*," (Princeton Univ. Press), 61-2.
5. Smith's book, published by the American Philosophical Society in 1999, is actually "*Transactions of the American Philosophical Society*," **89**, Part 3.
6. G. Sarton, "Introduction to the History of Science," (Baltimore 1927-47).
7. A. Mark Smith, "*Ptolemy's Theory of Visual Perception: An English Translation of the Optics with Introduction and Commentary. Transactions of the American Philosophical Society*," **86**, Part 2 (1996)
8. See, for example, O. Neugebauer "*A History of Ancient Mathematical Astronomy*," **I** (Springer-Verlag, 1975) 21.
9. Boyer says this, so does Smith in his 1999 book. See also Neugebauer's "*History of Ancient Mathematical Astronomy*," **II** (Springer-Verlag 1975), 894-6.
10. See, for instance, for an example from Babylon, J. Evans's "*The History and Practice of Ancient Astronomy*," (Oxford Univ. Press, 1998), 333.
11. Smith, 1999, page 8. See also his article "Ptolemy's Search for a Law of Refraction: A Case-Study in the Classical Methodology of 'Saving the Appearances' and Its Limitations," in Archive for History of Exact Sciences, **26** (1982) ," 221-40.
12. Seneca observed this phenomenon and tried to use it to argue his case for the rainbow. See his "*Natural Questions*," Section I, 6.5 , in T. Corcoran's translation in the Loeb Classical Library (Harvard Univ. Press, 1971) 54-7.
13. See Neugebauer (1975) **II**, 892-3.12.
14. See Boyer (1959) 110-25.
15. See, for example. R.A.R. Tricker's "*Introduction to Meteorological Optics*," 50-3. I leave derivation of the formulas for higher-order rainbows as an exercise for the reader. I note, with regret, that Ptolemy's results don't work so well for secondary rainbows, predicting that they lie at 35 degrees, rather than the true value of 50 degrees.