The Golden Ratio in Optics

BY BERTON C. WILLARD

Like the irrational numbers π and e, the golden ratio—itself an irrational number—keeps cropping up in the most diverse fields, from architecture to zoology. It was known in many forms, although not by the name golden ratio, to the mathematicians of the school of Pythagoras, the Greek philosopher (569-500 B.C.).

Perhaps the simplest source is in finding the golden cut of a straight line. If a line is cut into two lengths, 1 and x, so that $(1+x)/x = x/1$, then the golden cut has been made and x is known as the golden ratio. This ratio has fascinated mathematicians and others ever since, but it was not until the early part of the 20th century that the Greek letter Φ was suggested as its designation.

Numerically, the golden ratio is defined by:

$$\Phi=\frac{1+\sqrt{5}}{2}=1.6180339887 \quad ... \qquad (1)$$

It is also found in the Fibonacci series (after Leonardo Fibonacci, born 1175 A.D.), where each number in the series is the sum of the previous two, starting with 0 and 1:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... \qquad (2)$$

The ratio of any two adjacent numbers oscillates about and approaches Φ the farther one goes out in the series. As

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fascinating as this may sound, it is even more remarkable that this remains true for a series that starts with any two numbers and follows the same rule, giving the Lucas sequence, for example:

$$5, 2, 7, 9, 16, 25, 41, \ldots \tag{3}$$

Phi can also be found in the continued fraction made with the digit one:

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}} \tag{4}$$

Phi is the only number from which its reciprocal is obtained by subtracting 1, and its square is found by adding 1:

$$\Phi - 1 = \frac{1}{\Phi} \tag{5}$$

$$1 + \Phi = \Phi^2 \tag{6}$$

This is but a small sample of the ubiquitous Φ, and anyone who finds beauty in numbers and wishes to explore this further is encouraged to read *The Divine Proportion*, by Huntley.[1]

Phi in Optics

In geometric optics, Φ can be found hiding in several solutions to two-mirror telescope systems. Consider the Cassegrain telescope composed of a concave parabolic primary and a convex hyperbolic secondary, as shown in Fig-

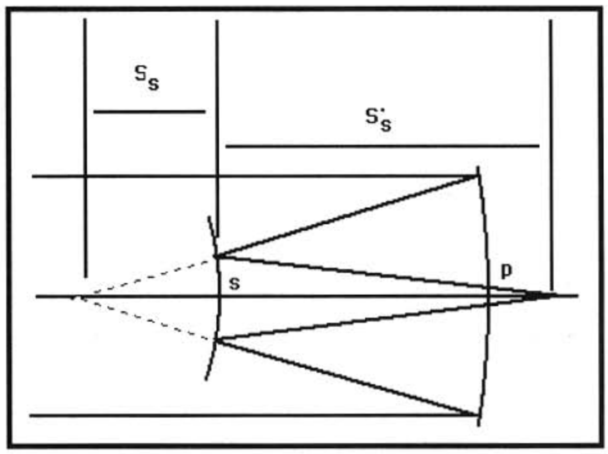

Figure 1. The Cassegrain telescope with paraboloidal primary and hyperboloidal secondary. If the secondary eccentricity is $\sqrt{5}$, then its amplification ratio, m, becomes Φ^2.

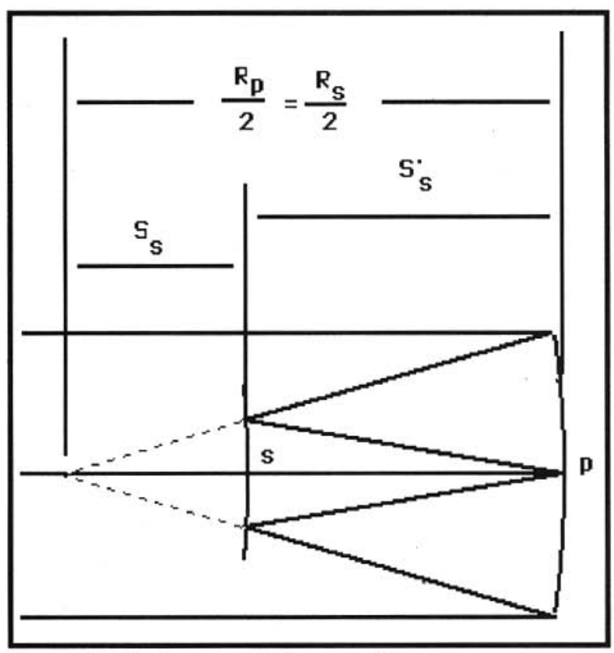

Figure 2. The Cassegrain telescope. If the mirror radii are equal and if the final focus is at the vertex of the primary, then the secondary amplification ratio, m, becomes Φ.

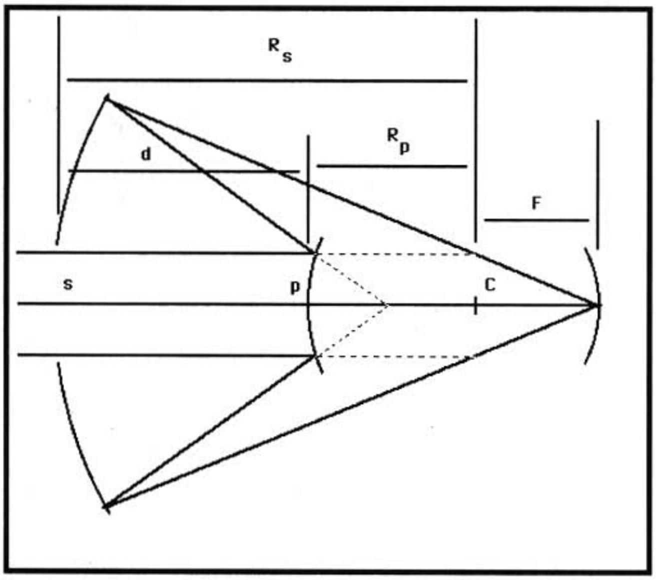

Figure 3. The Schwarzchild two-concentric spherical mirror telescope. The point p makes a golden cut of the line sc. Also, $d/R_p = R_s/d = F/F_p = F_s/\Phi = m = \Phi$.

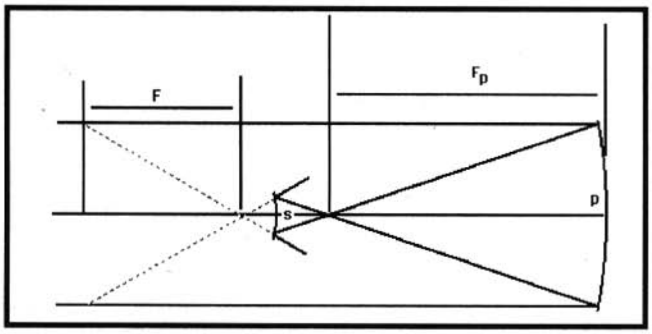

Figure 4. Two-concentric-sphere anastigmatic telescope with virtual image. In this case, $F_p/F = \Phi$.

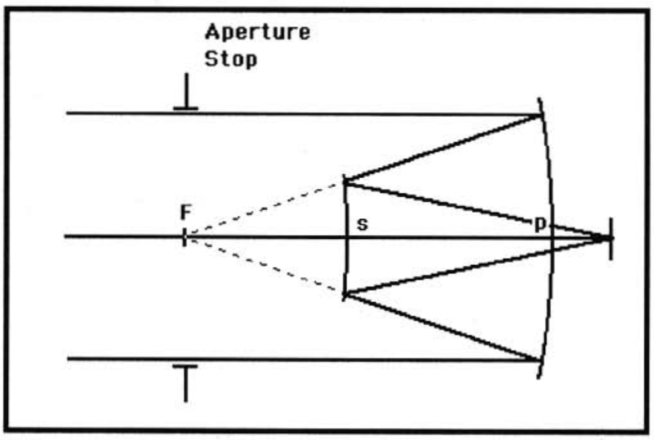

Figure 5. The Cassegrain telescope. Both mirrors have the same radius of curvature, are conic sections, and share one conic focus, while the aperture stop is at one conic focus. The eccentricities of both mirrors may be chosen such that there is no higher order field curvature, no third order coma, and no astigmatism of any order. In this case, the eccentricity of the primary is Φ and that of the secondary is $2+\sqrt{5}$.

ure 1. The secondary causes the telescope to have a focal length longer than that of the primary alone, where the amplification ratio, m, is given by Brueggemann:[2]

$$m=\frac{e+1}{e\cdot 1} \tag{7}$$

and e is the secondary eccentricity. If e is set equal to the square root of 5, m becomes Φ^2.

In another Cassegrain arrangement shown in Figure 2, if the radii of the two mirrors are made equal so the Petzval curvature is zero, and if the final focus is at the primary vertex, then $m = \Phi$, as mentioned by Brueggemann.[3]

Consider next the Schwarzschild two concentric sphere design of Figure 3, in which the primary is convex and the secondary is concave, with the object at infinity. Not only is this unique design corrected for all third order aberrations except Petzval curvature, it abounds in hidden values of Φ, as well. The point p in Figure 3, which is the primary vertex, makes a golden cut of the line sc, which is the radius of curvature of the secondary. That is:

$$\frac{R_S}{d}=\frac{d}{R_P}\,. \tag{8}$$

Also the following relationships exist:[4]

$$\frac{d}{R_P}=\Phi \qquad \frac{R_S}{d}=\Phi \qquad \frac{R_S}{R_P}=\Phi^2 \tag{9}$$

The third equation of (9) is mentioned by Kingslake,[5] although only in the form:

$$\frac{R_S}{R_P}=\frac{\sqrt{5}+1}{\sqrt{5}-1} \tag{10}$$

From equations (9), it follows that, as shown by Linfoot:[6]

$$F/F_P=\Phi \qquad F_S/F=\Phi \qquad F_S/F_P=\Phi^2 \tag{11}$$

From Figure 3, the magnification of the secondary, m, is given by:

$$m=\frac{R_S+F}{R_S-\frac{R_S}{2}} \tag{12}$$

It is straightforward to show that this reduces to $m = \Phi$, as pointed out by Wetherell and Rimmer,[7] although they also gave only the numerical value of 1.618 and did not mention the golden ratio.

The only other concentric two spherical mirror anastigmatic telescope has a concave primary and convex secondary, as shown in Figure 4. Unfortunately, the final image is virtual. However, the golden ratio is found in the following equations:[8]

$$F=\frac{\sqrt{5}-1}{2}(F_P)=\frac{F_P}{\Phi}, \tag{13}$$

where F = system focal length and F_p = primary focal length.

Figure 5 shows the final Cassegrain telescope to be considered. Both mirrors are conic, have the same radius of curvature, share one conic focus, and the aperture stop is at one of the conic focii. Because of equal radii, there is no Petzval curvature. The eccentricities of both mirrors may be chosen such that there is no higher order field curvature, no third order coma, and no astigmatism of any order.[9] Third order spherical aberration does exist. The eccentricities turn out to be:

$$e_P = \Phi \qquad e_S = 2+\sqrt{5}. \tag{14}$$

So far, all the cases presented are about two-mirror systems. Figure 6 shows a concentric–aplanatic lens. By making the first surface concentric to the aperture stop, there is no third order coma, astigmatism, or distortion contribution from that surface. If the second surface is aplanatic, there is no spherical aberration or coma of all orders, and no third order astigmatism. If the two radii are made equal so the Petzval curvature is zero, then the index of refraction must be equal to Φ.[10]

The last case involves a double window with three reflecting surfaces,[11] as shown in Figure 7. A ray enters the window from the left. How many rays will emerge from the window after allowing 0, 1,2,3,4,... reflections? The answer is the Lucas sequence 1,2,3,5,8,....

Most likely, other Φs can be found lurking in little-known theorems of optics. It is the author's hope that they, too, will be published.

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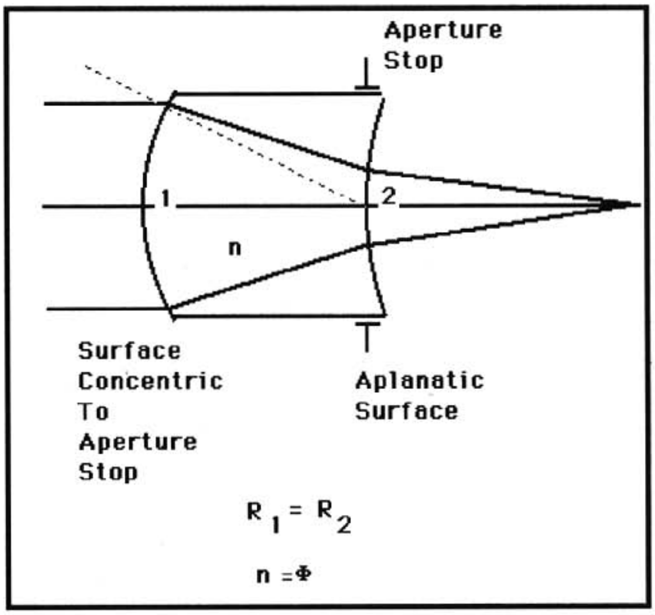

Figure 6. Concentric-aplanatic lens. If the radii are equal so the Petzval curvature is zero, then the index of refraction is Φ.

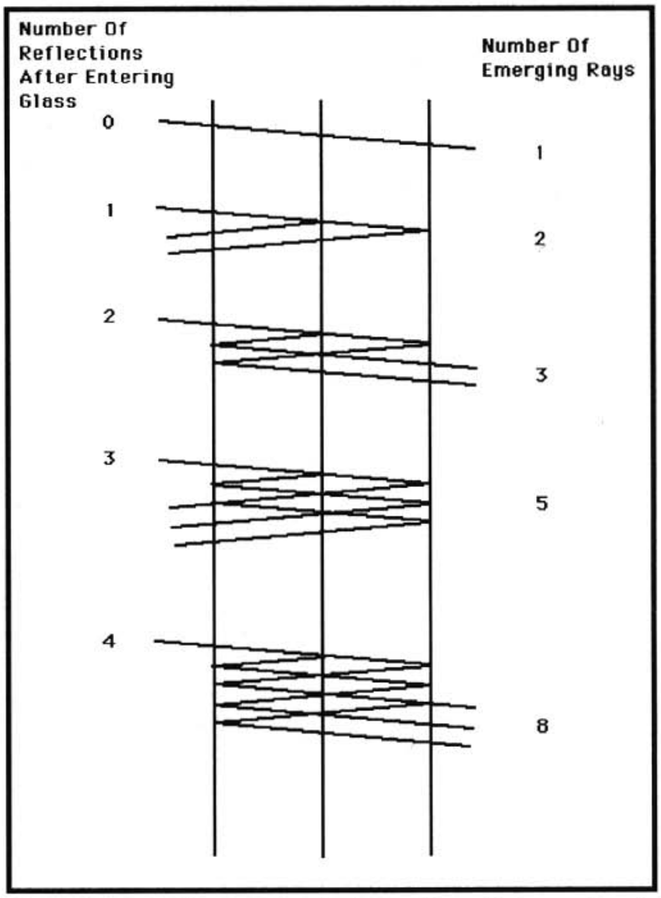

Figure 7. A double window with three surfaces. If the number of reflections allowed for any ray after entering the glass is increased as 0,1,2,3,4...., then the total number of rays emerging is the Lucas sequence 1,2,3,5,8,....